A SIMPLE 4-DIMENSIONAL NONFACET*

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ABSTRACT

An infinite family of simple (i.e. 4-valent) 4-dimensional convex polytopes is constructed with the property that no 5-dimensional convex polytope has each of its 4-dimensional faces combinatorially equivalent to just one member of this family.

1. **Introduction.** A convex d-polytope P is a *d-nonfacet* provided there is no $(d+1)$-polytope all of whose facets are combinatorially equivalent to P. Perles and Shephard were the first to discover the existence of nonfacets of dimensions greater than two [3]. They constructed many nonfacets of all dimensions but they were not able to find any simple (d-valent) d-nonfacets of dimensions greater than 3. In this paper we show the existence of an infinite number of simple 4-nonfacets.

2. **Defiinitions.** If P is a d-polytope then by a k-manifold M in P we will mean a strongly connected k-dimensional subcomplex of the boundary complex, $\beta(P)$, of P such that each $(k-1)$-cell in M meets exactly two k-cells of M. By an s-manifold in P we will mean a $(d-2)$-manifold in P which is (topologically) a $(d-2)$-sphere. A *Hamiltonian manifold* is an s-manifold which contains all vertices of P.

Our construction is accomplished by means of a *Joining process* which we now define. Let P and Q be two simple d-polytopes and let x and y be vertices of P and Q respectively. The two polytopes are joined by performing the following steps:

(i) Truncate vertices x and y producing polytopes P_1 and Q_1 with simplicial facets F_1 and F_2, respectively, which were created by the truncation.

(ii) Take a hyperplane, H passing through x and apply a projective trans-

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formation τ_1 which sends H to infinity. In $\tau_1(P_1)$ all facets meeting $\tau_1(F_1)$ will be parallel. Apply the same kind of transformation τ_2 to Q_1.

(iii) Apply an affine transformation α_1 to $\tau_1(P_1)$ which will produce a polytope $P_2 = \alpha_1[\tau_1(P_1)]$ in which one facet meeting $\alpha_1[\tau_1(F_1)]$ is perpendicular to $\alpha_1[\tau_1(F_1)]$. Note that all facets meeting $\alpha_1[\tau_1(F)]$ will be perpendicular to it. Apply the same kind of affine transformation, α_2 to $\tau_2(Q_1)$ to prodcue $Q_2 = \alpha_2[\tau_2(Q_1)]$.

(iv) Apply an affine transformation α_3 to P_2 which will take $\alpha_1[\tau_1(F_1)]$ onto $\alpha_2[\tau_2(F_2)]$ and leaves the faces meeting $\alpha_1[\tau_1(F_1)]$ perpendicular to it.

(v) Place Q_2 and $\alpha_3(P_2)$ so that $\alpha_3[\alpha_1(\tau_1(F_1))]$ and $\alpha_2[\tau_2(F_2)]$ coincide and so that the interior of Q_2 misses the interior of $\alpha_3(P_2)$.

This joining process produces a simple polytope which we shall denote by $P + Q$.

Since we are concerned only with the combinatorial structure of polytopes we shall, from now on, not distinguish between the various images of our polytopes under affine and projective transformations.

3. **Manifolds in polytopes.** In order to obtain our result we shall use the following theorem by Perles and Shephard [3].

If a d-polytope P is a facet (i.e. *not a nonfacet*) *then*

$$f_j(P) < m_j(P)(d + 1 - j)/(d - 1 - j).$$

Here, $f_j(P)$ is the number of f-faces of P and $m_j(P)$ is the maximum number of j-faces that occur among all regular projections of all d-polytopes combinatorially equivalent to P. A regular projection of P is defined as follows. Let x be a vector which is not parallel to any face of P then the $(d - 1)$-polytope obtained by an orthogonal projection of P onto a hyperplane normal to x is called a *regular projection* of P.

In a regular projection Q of any 4-polytope P the 2-faces of Q are images of 2-faces in P and thus the boundary complex of Q is the image of an s-manifold in P. We shall use (1) for the case $j = 1$ in which case

$$f_1(P) < \frac{4}{2} m_j(P). \tag{2}$$

In view of (2) and the above remarks it is sufficient to construct an infinite number of simple 4-polytopes in which each s-manifold uses at most one half of the vertices of the polytope.

LEMMA 1. *If M is a $(d-2)$-manifold which is a subcomplex of the boundary complex of a d-simplex, $d \geqq 3$ then M is a topological sphere.*

Proof. We shall prove the lemma by induction on the dimension, d, of the simplex. The result is obvious for $d = 3$. Suppose M is a manifold in a d-simplex S, $d > 3$, and that the lemma is true for all dimensions $3 \geqq d' > d$.

Let B be the union of all facets of S lying inside M. Since at most d facets lie inside M, they intersect on some k-face, F, $0 > k > d$. We shall show that in constructing B by adding one facet at a time to F we will create a new $(d-1)$-cell at each step. This is clearly true when we add the first facet. Suppose we add the ith facet $\mathscr{F}$, $i > 1$. The intersection of $\mathscr{F}$ with the previously constructed cell consists of a collection of $(d-2)$-faces of $\mathscr{F}$. Since F is also a simplex these $(d-2)$-faces intersect on some face of $\mathscr{F}$. By induction the boundary $\beta(C)$ of the union, C of this collection of $(d-2)$-facets is a $(d-3)$-sphere and C is a $(d-2)$-cell. By adding the facet $\mathscr{F}$ to the previously constructed cell we have altered the boundary $\beta(C)$ by replacing a $(d-2)$-cell in $\beta(C)$ by the complement of C in the boundary of $\mathscr{F}$, which is also a $(d-2)$-cell, thus by adding $\mathscr{F}$ we have created a new $(d-1)$-cell. Thus the union of all facets of S which lie inside M is a cell and M is a sphere.

LEMMA 2. *Let P and Q be simple d-polytopes, $d \geqq 4$ and M be an s-manifold in $P + Q$. If M contains vertices of both P_1 and Q_1, then there is an s-manifold M_1 in Q such that:*

(i) *M_1 contains each vertex common to Q_1 and M.*

(ii) *M_1 contains the vertex y of Q which was truncated in the joining process.*

Proof. Let H be the hyperplane determined by the facet F_2. By our construction, each facet of $P + Q$ which meets F_2 is perpendicular to H, thus each $(d-2)$-face of M which intersects H, intersects F in a $(d-3)$-face of F_2.

Each $(d-3)$-face of M is the intersection of exactly two $(d-2)$-faces of M, thus in $M \cap H$ each $(d-4)$-face is the interesection of exactly two $(d-3)$-faces. This shows that the connected components of $M \cap H$ are manifolds in F_1. If there were two components of $M \cap H$ then they would separate at least two facets of F_2, but since F_2 is a simplex, each pair of facets meet and thus $M \cap H$ has only one component. By Lemma 1 this component is a sphere. Let C be one of the regions into which $\beta(F_2)$ is divided by $M \cap H$. The sets C and $M \cap Q_1$ are $(d-2)$-cells which have a common boundary and thus they form an s-manifold

M' in Q_1. Let S be the d-simplex containing y which was created by the truncation. We shall place S on Q_1 (after applying the suitable transformations) to produce Q. Let C^* be the $(d-2)$-cell composed of $(d-2)$-faces of S which coincide with the $(d-2)$-faces of C. Let B be the union of the sets of the form con $([v] \cup F)$ where F is a $(d-2)$-face of C^*. The set B is a $(d-1)$-cell and its boundary, $\beta(B)$ is a $(d-2)$-sphere. If we now take the union of $M' \sim C$ and the complement of C^* in $\beta(B)$ we will have our desired s-manifold.

LEMMA 3. *The* 120-*cell contains no Hamiltonian manifold.*

Proof. The 120-cell is a simple 4-polytope with 120 regular dodecahedral facets and 600 vertices [1]: Suppose M were a Hamiltonian manifold in the 120-cell. Each 2-face of M would be a pentagon, thus $E = 5E/2$ where E is the number of edges of M and F is the number of 2-faces. Euler's formula for M states that $V - E + F = 2$, where V is the number of vertices of M. This implies that $600 - 3E/5 = 2$, which admits no integer solution for E.

THEOREM 1. *There exists a sequence $P_0, P_1, \cdots$ of simple 4-polytopes for which $p_k < n_k^{1-\beta}$, where P_k is the maximum number of vertices in any s-manifold in P_k, n_k is the number of vertices of P_k and β is a positive constant.*

Proof. The 3-dimensional analogue of this theorem has been proved by Grünbaum and Motzkin [2] and our proof is almost identical to theirs. We define P_1 to be the 120-cell and we shall define P_i inductively. If n_{i-1} is the number of vertices of P_{i-1} then we form P_i by joining n_{i-1} copies of P_{i-1} to a copy of P_{i-1} (one copy to each vertex). This gives us a new polytope P_i with $n_{i-1}(n_{i-1}-1)$ vertices. If p_i is the maximum number of vertices in any manifold in P_i then $p_i < p_{i-1}(p_{i-1}-1)$. We now choose $\beta > 0$ such that

$$\frac{p_0}{n_0} < n_0^{-\beta}$$

and proceeding by induction we show that

$$\frac{P_{k+1}}{n_{k+1}} < n_{k+1}^{-\beta} .$$

This will follow because:

$$P_k < n_k \quad \text{and we have} \quad \frac{P_k - 1}{n_k - 1} < \frac{P_k}{n_k}.$$

Thus

$$\frac{P_{k+1}}{n_{k+1}} < \frac{P_k(p_k-1)}{n_k(n_k-1)} < \frac{p_k^2}{n_k} < n_k^{-2\beta} .$$

since $n_{k+1} = n_k(n_k-1) < n_k^2$, we have

$$n_k^{-2\beta} < n_{k+1}^{-\beta} \quad \text{and the result follows.}$$

Using the above theorem we may find an infinite collection of simple 4-polytopes for which $p_k < n_k/2$ and using the Perles-Shephard Theorem we have an infinite collection of simple 4-dimensional nonfacets.

4. **Remarks,**

(1) One can prove that if P is the dual of a cyclic 4-polytope and has more than 8 facets then it has no s-manifold, thus one could use any of these polytopes instead of the 120-cell. The author conjectures that similar results hold for the duals of even-dimensional cyclic polytopes of higher dimensions.

(2) The author is indebted to the referee for his suggested proof of the inequality in Theorem 1.

References

1. H. S. M. Coxeter, *Regular Polytopes*, p. 157, London, 1948.

2. B. Grünbaum and T. S. Motzkin, *Longest simple paths in polyhedral graphs*, J. London Math. Soc. **37** (1962), 152–160.

3. M. Perles and G. Shephard, *Facets and nonfacets of convex polytopes*, Acta. Math. **119** (1967), 113–145.

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